MS-R-7211

AD735785

Asymptotic Expansions of Integral Transforms With Oscillatory Kernels: A Generalization of the Method of Stationary Phase

by

R. A. Handelsman* and N. Bleistein†

1971

JAN 26 1972

D

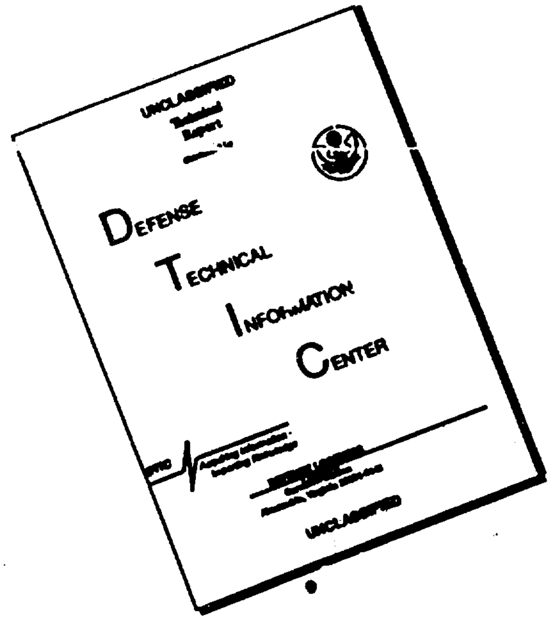
*University of Illinois, Chicago Circle Campus

†University of Denver

This research was partially supported by the Air Force Office of Scientific Research under grant AFOSR 70-1923 and partially by the National Science Foundation.

Approved for public release; distribution unlimited.

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of Denver Dept of Mathematics Denver, Colorado 80210	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

ASYMPTOTIC EXPANSIONS OF INTEGRAL TRANSFORMS WITH OSCILLATORY KERNELS: A GENERALIZATION OF THE METHOD OF STATIONARY PHASE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (First name, middle initial, last name)

R. A. Handelsman
N. Bleistein

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1971	36	9

8a. CONTRACT OR GRANT NO. AFOSR-70-1923	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 9749	
c. 61102F d. 681304	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) TR-72-0171

10. DISTRIBUTION STATEMENT

Approved for public release;
distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
TECH, OTHER	Air Force Office of Scientific Research (NM) 1400 Wilson Blvd Arlington, Virginia 22209

13. ABSTRACT

Integrals with integrands of the form $H(\lambda\phi(t))\,f(t)$ are considered for $\lambda \to \infty$ and $H(t)$ oscillatory for large argument. It is shown that the set of critical points for such integrals includes zeros of the phase function ϕ as well as all of those that arise in the analysis of the standard integrals of Fourier type; i.e., for the special case where $H(t) = \exp(it)$. The contribution to the asymptotic expansion from each type of critical point is derived. In particular, a formula is obtained which generalizes the stationary phase formula associated with Fourier type integrals.

DD FORM 1 NOV 65 1473

Security Classification

1. Introduction.

In the method of stationary phase, one is concerned with the asymptotic expansion, as $\lambda \to \infty$, of functions defined by integrals of the form

$$(1.1) \qquad I(\lambda) = \int_a^b \exp\{i\lambda\sigma(t)\}f(t)dt.$$

We shall assume that the details of this method are familiar to the reader and need not be discussed.

Our concern here shall be with the asymptotic expansion of integrals of the form

$$(1.2) \qquad I(\lambda) = \int_a^b H(\lambda\sigma(t))f(t)dt$$

in the case where the kernel function $H(t)$ is oscillatory for both large positive and large negative arguments. More precisely, we assume that, as $t \to +\infty$,

$$(1.3) \quad H(t) \sim \exp\left\{i \sum_{\ell=0}^{\ell<\nu/\delta} b_\ell t^{\nu-\delta\ell}\right\} \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} t^{-r_m}(\log t)^n.$$

Here each b_ℓ is real, $N(m)$ is finite for all m and $\{Re(r_m)\}$ is a monotonically increasing sequence with limit $+\infty$. In the limit $t \to -\infty$, we assume that an expansion of the form (1.3) holds with t replaced by $|t|$ and with, in general, different constants.

Clearly the Fourier kernel $\exp\{it\}$ is a special case of the general kernel we propose to study. Thus

we should expect to recover from our asymptotic analysis of (1.2) the stationary phase results valid for (1.1).

We should point out that there are functions, such as the Airy function $Ai(t)$, which are oscillatory in one of the limits $t \to \pm\infty$ and exponential in the other. It will be apparent that integrals (1.2) with such functions as kernels can also be treated by the methods to be developed below.

The method itself involves applications and generalizations of an asymptotic technique recently developed by Handelsman and Lew [1], [2], [3]. This technique makes heavy use of the Mellin transform whose relevant properties are discussed below. In references [1], [2] and [3] only the case $\varphi(t) = t$ is treated. Here, however, we shall consider more general φ and, in particular, shall allow φ to be non-monotonic.

As is well-known, to derive the asymptotic expansion of (1.1) one must first identify its set of critical points. These include end points of integration, stationary points of φ and points where either φ or g fails to be infinitely differentiable. We shall show that, in addition to all of the above, the set of critical points for (1.2) also includes the zeros of φ. Indeed, this is one of the main results of this paper.

To gain some insight into the critical nature of

the zeros of φ, suppose that (1.3) is only an asymptotic result and, in particular, is not valid near $t = 0$. Then, no matter how large λ is, there always exists a neighborhood of each zero of φ throughout which $H(\lambda\varphi)$ is not asymptotically described by our assumed asymptotic forms. In other words, the asymptotic expansion of $H(\lambda\varphi)$, as $\lambda \to \infty$, undergoes a drastic change as t passes through any of these neighborhoods. For this reason we can think of these neighborhoods as "boundary layer regions." It is certainly reasonable that the rapid change in the asymptotic behavior of $H(\lambda\varphi)$ as t passes through a boundary layer region will affect the asymptotic expansion of I.

In the light of the above argument we can understand why the zeros of φ are not critical points for (1.1). Indeed, the Fourier kernel $H(t) = \exp(it)$, has an asymptotic expansion as $t \to +\infty$ of the form (1.3). This expansion, however, holds for all t so that there are no boundary layer regions of the type just described.

In the following section we reduce our problem to the study of certain integrals of canonical type. In Section 3, we consider some results concerning Mellin transforms that are needed to implement our methods. Finally the desired asymptotic expansion of I is

obtained in Sections 4 and 5.

2. Reduction to Canonical Integrals.

Because there are many possible critical points for integrals of the form (1.2) with H an oscillatory kernel, it is convenient to have a means for isolating them so that their contributions to the asymptotic expansion of I can be studied separately. This can be accomplished by using neutralizer functions. These were first introduced by Van der Corput [4], and we shall assume that their basic properties are familiar to the reader. The net effect of the neutralization process is to reduce the asymptotic analysis of (1.3) to the study of a sum of integrals each having exactly one critical point either as an upper or as a lower end point of integration.

Suppose first that $t = t_0$ is a critical point at which φ is non-zero. After neutralization, $t = t_0$ will appear as either an upper or lower end point of integration in at most two of the integrals to be asymptotically evaluated. To obtain the corresponding contributions to the asymptotic expansion of I, one need only replace $H(\lambda\varphi)$ by the appropriate asymptotic expansion and integrate the resulting series term by term. Thus, finding these contributions is reduced to

the asymptotic evaluation of many integrals of the form (1.1). We have, therefore, that the only critical points which require non-standard methods of analysis are the zeros of ϖ.

As a result of the above discussion, we shall focus our attention on obtaining the contribution to the asymptotic expansion of (1.2) corresponding to a given zero of ϖ. If we denote this zero by $t = c$ and the contribution by $I_c(\lambda)$, then after neutralization we find that

$$I_c(\lambda) = I_{c_+}(\lambda) + I_{c_-}(\lambda) \tag{2.1}$$

where

$$I_{c_-}(\lambda) = \int_a^c H(\lambda\varpi)g_{c_-}(t)dt \tag{2.2}$$

and

$$I_{c_+}(\lambda) = \int_c^b H(\lambda\varpi)g_{c_+}(t)dt. \tag{2.3}$$

Here $g_{c_-}(t)$ vanishes for $t < \alpha < c$ with α chosen so that ϖ vanishes in $[\alpha,c]$ only at $t = c$ and ϖ' does not vanish in $[\alpha,c)$. Furthermore, $g_{c_-}(t) = g$ in some small half neighborhood of $t = c_-$.

Similarly, $g_{c_+}(t)$ vanishes for $c < \beta < t$ with β chosen so that ϖ vanishes in $[c,\beta]$ only at $t = c$ and ϖ' does not vanish in $(c,\beta]$. Finally, $g_{c_+} = g$

in some small half neighborhood of $t = c_+$. Of course if c coincides with one of the end points of integration in (1.2), then only one of the integrals I_{c_-}, I_{c_+} is non-zero.

Suppose now that as $t \to c_-$, ϖ has an asymptotic expansion whose leading term is given by

$$\varpi \sim \gamma_0 (c-t)^{\nu_0}, \qquad \nu_0 > 0. \tag{2.4}$$

If in (2.2) we introduce the new variable of integration

$$s = \mu_{c_-} \varpi(t), \qquad \mu_{c_-} = \operatorname{sgn} \gamma_0 \tag{2.5}$$

then we can write

$$I_{c_-}(\lambda) = \int_0^\infty H(\lambda \mu_{c_-} s) G_-(s) ds. \tag{2.6}$$

Here

$$G_- = g_{c_-}(t(s)) \frac{dt}{ds} \tag{2.7}$$

which we note vanishes for $s > \mu_{c_-} \varpi(\beta)$.

Similarly, if we assume that as $t \to c_+$

$$\varpi(t) \sim r_0 (t-c)^{\rho_0}, \qquad \rho_0 > 0, \tag{2.8}$$

and set

$$s = \mu_{c_+} \varpi, \qquad \mu_{c_+} = \operatorname{sgn} r_0, \tag{2.9}$$

then I_{c_-} can be written

$$I_{c_+}(\lambda) = \int_0^\infty H(\lambda\mu_{c_+}s)G_+(s)ds. \tag{2.10}$$

Here

$$G_+(s) = g_{c_+}(t(s))\frac{dt}{ds} \tag{2.11}$$

which vanishes for $s > \mu_{c_+}\varphi(\alpha)$.

Thus we have reduced our problem to the study of the two canonical integrals

$$I_\pm(\lambda) = \int_0^\infty H(\pm\lambda s)G(s)ds \tag{2.12}$$

where the kernel $H(t)$ is oscillatory in each of the limits $t \to \pm\infty$ and $G(s)$ vanishes for s outside of some finite interval.

3. Results on Mellin Transforms.

As we shall see, Mellin transforms play an important role in our asymptotic development. Indeed, one might anticipate this upon observing that each of the canonical integrals (2.12) can be expressed as a Mellin convolution [3].

The Mellin transform of a function $f(s)$ is defined by

$$M[f;z] = \int_0^\infty f(s)s^{z-1}ds, \qquad z = x+iy \tag{3.1}$$

when this integral exists. Furthermore if f is such that

$$f(s) = O(s^p), \qquad s \to 0+ \tag{3.2}$$
$$f(s) = O(s^{-r}), \qquad s \to +\infty.$$

Then $M[f;z]$ converges and is holomorphic in the strip

$$-p < \mathrm{Re}(z) = x < r. \tag{3.3}$$

Also, within this strip $\lim_{|y| \to \infty} |M[f;x+iy]| = 0.$

If in (3.2) $-p > r$, then $M[f;z]$ does not exist in the ordinary sense. Nevertheless with certain additional assumptions on f one can define $M[f;z]$ in a generalized sense [3]. We shall not need to appeal to this extension in this paper however.

From our point of view, there are two results concerning Mellin transforms that are of special significance. The first is the simple relation

$$M[f(\lambda s);z] = \lambda^{-z}\int_0^\infty f(s)s^{z-1}ds = \lambda^{-z}M[f;z]. \tag{3.4}$$

The second involves integrals of the form

$$J = \int_0^\infty f(s)h(s)ds. \tag{3.5}$$

Indeed, suppose that $M[f;z]$ and $M[h;z]$ are holomorphic in overlapping vertical strips. (This will always be the case if J is absolutely convergent.) If $\mathrm{Re}(z) = c$ lies in the common strip of analyticity, then we have

$$(3.6) \qquad \int_0^\infty f(s)h(s)ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[h;z]M[f;1-z]dz$$

which is Parseval's Theorem for Mellin transforms.

Upon combining these last two results, we find that our canonical integrals (2.12) have the representations

$$(3.7) \qquad I^{\pm}(\lambda) = \frac{1}{2\pi i}\int_{c_\pm - i\infty}^{c_\pm + i\infty} \lambda^{-z} M[H(\pm s);z]M[G(s);1-z]dz.$$

Here $Re(z) = c_\pm$ lies in the strip of analyticity of the integrand. We note that the total dependence of the integrand on λ is contained in the factor λ^{-z}.

Our plan is to push the contour of integration in (3.7) to the right, apply Cauchy's integral theorem, and derive thereby an asymptotic expansion of $I^{\pm}(\lambda)$ as a residue series. In order to accomplish this we must obtain certain information about the analytic continuations of the functions $M[H(\pm s);z]$ and $M[G(s);1-z]$ into the right half plane. Specifically, we must locate and classify the singularities of these continuations, and we must estimate their behavior as $z \to \infty$ along vertical lines.

For oscillatory functions, the required information is contained in

Lemma 1. Suppose that $H(s)$ is locally integrable on $(0,\infty)$ and satisfies (3.2) with $-p < r$. Suppose further

that, as $s \to +\infty$, $H(s)$ has an asymptotic expansion of the form (1.3) in which event $-r = r_0$. Then $M[H;z]$ can be continued into the right half plane $Re(z) > -p$ as a holomorphic function. Furthermore, in this right half plane

$$(3.8) \qquad |M[H;z]| = O\left(|y|^{(x-Re(r_0))/\nu - \frac{1}{2}}\right)$$

as $|y| \to \infty$.

<u>Proof</u>: The proof of this lemma is given in the Appendix.

As an example and to illustrate the sharpness of the estimate (3.8), let us consider the function $H = \exp(is^\nu)$ whose Mellin transform is given by [5]

$$(3.9) \qquad M[\exp(is^\nu);z] = \exp\left(\frac{i\pi z}{2\nu}\right)\Gamma\left(\frac{z}{\nu}\right), \qquad 0 < x < 1.$$

In this case $r_0 = 0$, and the analytic continuation into $Re(z) > 1$ is explicit. We note that $\Gamma\left(\frac{z}{\nu}\right)$ is analytic in $Re(z) > 0$. The estimate (3.8) follows from the known asymptotic expansion of the gamma function

$$(3.10) \qquad \Gamma(z) \sim |y|^{x-\frac{1}{2}}\exp[-\pi|y|/2], \qquad |y| \to \infty.$$

As we have indicated above, our plan is to push the contour of integration in (3.7) to the right. In order to accomplish this we must of course determine

the analytic continuation of $M[G;1-z]$ into a right half plane. Let us assume for the present that this has been done and $M[G;1-z]$ is a meromorphic function. Then to justify the displacement of the contour to the line $\mathrm{Re}(z) = k > c_\pm$, one must still show that

$$(3.11)\quad \lim_{|y|\to\infty} M[H(\pm s);z]M[G(s);1-z] = 0, \quad c_\pm \leq x \leq k.$$

The estimate (3.8) implies an algebraic growth of $M[H(\pm s);z]$ in this limit which worsens with increasing x. This growth must therefore be compensated by a commensurate decay of the analytic continuation of $M[G(s);1-z]$. In the following sequence of lemmas, we shall establish sufficient conditions for such decay.

Lemma 2. Let $G(s)$ be q times continuously differentiable on $(0,\infty)$. Let $G^{(q+1)}(s)$ be piecewise continuous on $[0,k]$ and continuous for $s \geq k$. Finally suppose that there exists a real number x_0 such that for all $x > x_0$, $\left(s\frac{d}{ds}\right)^p (s^x G(s))$ vanishes, as $s \to 0+$, for $p = 0,1,2,\ldots,q$ and, as $s \to \infty$, for $p = 0,1,2,\ldots,q+1$. Then, as $|y| \to \infty$,

$$(3.12)\qquad M[G;z] = O(|y|^{-q-1})$$

for all $x > x_0$.

Proof: The proof of this lemma is given in the Appendix.

Remarks: The hypotheses of Lemma 2 simply provide

sufficient information to allow for the estimation of $M[G;z]$ via integration by parts. Furthermore the assumptions on G imply that $M[G;z]$ is holomorphic in $-x_0 < Re(z)$.

The next two lemmas follow from analogous results for Fourier transforms. Their proofs will be omitted here, but can be constructed from the corresponding proofs in Titchmarsh [6].

Lemma 3. Let $G(s)$ satisfy the conditions of Lemma 2 except now replace the condition on $G^{(q+1)}(s)$ by the assumption that $\left(s\frac{d}{ds}\right)^q (s^x G(s))$ is of bounded total variation. Then

$$M[G;z] = O(|y|^{-q-1}), \tag{3.13}$$

as $|y| \to \infty$, for all $x > x_0$.

Lemma 4. Let $G(s)$ satisfy the conditions of Lemma 2 except now replace the condition on $G^{(q+1)}(s)$ by the assumption that $\left(s\frac{d}{ds}\right)^q (s^x G(s))$ is Hölder continuous of order γ on $[0,k]$ and of bounded total variation for $s > k$. Then

$$M[G;z] = O(|y|^{-q-\gamma}) \tag{3.14}$$

as $|y| \to \infty$, for all $x > x_0$.

Lemmas 2-4 yield estimates on the decay of $M[G;z]$ in its region of absolute convergence or equivalently in its region of analyticity. We now wish to obtain analogous information outside of this region. As we shall soon see, the analytic continuation of $M[G;z]$ to the left, (and hence of $M[G;1-z]$ to the right), depends to a large extent on the nature of $G(s)$ near $s = 0+$. Indeed we have the following result due to Handelsman and Lew [3]:

Lemma 5. Suppose that $M[G;z]$ is holomorphic in the region $-\alpha < \mathrm{Re}(z) < \beta$, and that, as $s \to 0+$,

$$G(s) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} d_{mn} s^{a_m} (\log s)^n \tag{3.15}$$

with $\mathrm{Re}(a_m) \uparrow \infty$ and $N(m)$ finite for each m. Then, $\alpha = \mathrm{Re}(a_0)$ in (3.2) and $M[G;z]$ can be continued into $\mathrm{Re}(z) \leq -\mathrm{Re}(a_0)$ as a meromorphic function with poles at the points $z = -a_m$. Moreover, about these points, $M[G;z]$ has a Laurent expansion with singular part

$$-\sum_{n=0}^{N(m)} d_{mn} \frac{\Gamma(n+1)}{(-z-a_m)^{n+1}}$$

Remark. We note that when $N(m) = 0$ for each m, i.e., when no logarithms appear in the expansion (3.15), all of the poles in the analytic continuation of $M[G;z]$ to the left are simple.

If we combine the results of Lemmas 1 and 5, then we can conclude that when $G(s)$ has an expansion, as $s \to 0+$, of the form (3.15), all of the singularities of the analytic continuation of $M[H(\pm s);z]$, $M[G;1-z]$ into the right half plane are determined by the exponents a_m. Moreover, these singularities are poles so that our proposed deformation of contour will indeed yield a residue series for $I^{\pm}(\lambda)$. We must still estimate $M[G;1-z]$ as $|y| \to \infty$ in order to justify the deformation. For this purpose we now state

Lemma 6. Let $G(s)$ satisfy the smoothness conditions of Lemma 2. Also, let $\left(s\frac{d}{ds}\right)^p (s^x G)$ vanish, as $s \to +\infty$, for $p = 0,1,\ldots,q+1$ and $x > 1-\mathrm{Re}(a_0) = x_0$. Finally suppose that (3.15) holds and that the asymptotic expansion of $G^{(m)}(s)$, $m = 0,\ldots,q+1$, as $s \to 0+$, is obtained by successively differentiating (3.15) term by term. Then

$$M[G;z] = O(|y|^{-q-1}), \qquad |y| \to \infty, \tag{3.16}$$

for all x. Here by $M[G;z]$ we mean the analytic continuation of this Mellin transform into the entire z-plane.

Proof: The proof of this lemma is given in the Appendix.

Corollary. If, in Lemma 6, the stated conditions hold

for all q, then $M[G;z] = O(|y|^{-r})$ for all r and all x.

We remark that, if in Lemma 6, the smoothness conditions of Lemma 2 are replaced by those of either Lemma 3 or Lemma 4, then the corresponding changes must be made in the estimate (3.16). Nevertheless, one still finds that the results obtained are valid for the analytic continuation of the Mellin transform into the entire complex plane.

To illustrate some of the results obtained above let us consider an explicit example. Indeed, suppose that

$$G(s) = s^a e^{-s} \tag{3.17}$$

which satisfies the conditions of Lemma 6 with $a_0 = a$ and $q = \infty$. Then the corollary predicts that, as $|y| \to \infty$, $M[G;z]$ decays faster than any power of $|y|$. For this example we have the explicit result

$$M[G;z] = \Gamma(a+z) = O\left(\exp\left(\frac{-\pi|y|}{2}\right)\right), \quad |y| \to \infty$$

which agrees with this prediction. Furthermore we have from known properties of the gamma function that the analytic continuation of $\Gamma(a+z)$ into $\operatorname{Re}(z) \leq -a$ has simple poles at the points $z = -(a+m)$, $m = 0,1,2,\ldots$, with corresponding singular parts

$$(3.18) \qquad (-1)^m/m!(z+a+m).$$

As is readily seen, this last result is in agreement with that predicted by Lemma 5.

4. Asymptotic Expansion of $I^{\pm}(\lambda)$.

By using the theory of Mellin transforms developed in the previous section, we shall now derive asymptotic expansions for the two canonical integrals (2.11). We first note that, if $H(\pm s)$ has the asymptotic expansion (1.3) as $s \to \infty$, and $G(s)$ has the asymptotic expansion (3.15) as $s \to 0+$, then in (3.7)

$$(4.1) \qquad c_{\pm} < \text{Min}(\text{Re}(r_0), \text{Re}(1+a_0))$$

since this is the right limit of the common strip of analyticity of the integrand in that equation.

We state the main result concerning the asymptotic behavior of $I^{\pm}(\lambda)$ in

Theorem 1. Let $G(s)$ satisfy the conditions of Lemma 6 and $H(\pm s)$ satisfy the conditions of Lemma 1. Then

$$(4.2) \qquad I^{\pm}(\lambda) = \sum_{\text{Re}(a_m+1)<k} \sum_{n=0}^{N(m)} d_{mn}\lambda^{-(a_m+1)} \cdot \sum_{j=0}^{n} \binom{n}{j}(-\log \lambda)^j M^{(n-j)}[H(\pm s);1+a_m] + \mathcal{E}(\lambda;k).$$

Here

$$(4.3)\quad \mathcal{E}(\lambda;k) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \lambda^{-z} M[H(\pm s);z]M[G(s);1-z]$$
$$= o(\lambda^{-k}), \qquad \lambda \to \infty$$

and

$$(4.4)\qquad k < \nu(q+\tfrac{1}{2}) + \mathrm{Re}(r_0)$$

where $k \neq \mathrm{Re}(a_m)+1$ for any m.

Proof: In the exact representation (3.7) we displace the vertical contour of integration to the line $\mathrm{Re}(z) = k > c_\pm$. We note that by Lemmas 1 and 5 the analytic continuation of the integrand in (3.7) into the right half plane is a meromorphic function with poles at the points $z = a_m+1$, $m = 0,1,\ldots.$ Indeed, we find, upon formally applying Cauchy's integral theorem, that (4.2) is valid. Thus to complete the proof of the theorem we need only justify the displacement itself and establish the error estimate given by (4.3) and (4.4).

It follows from (3.8) and (3.16) that

$$(4.5)\quad M[H(\pm s);x+iy]M[G(s);1-x-iy] = O(|y|^{-\varepsilon(x)}), \qquad |y| \to \infty.$$

Here

$$(4.6)\qquad \varepsilon(x) = q + \tfrac{3}{2} - (x-\mathrm{Re}(r_0))/\nu.$$

Thus we can displace the contour to the line $\mathrm{Re}(z) = k$ so long as $\varepsilon(k) > 0$, i.e., so long as

$$(4.7) \qquad k < \nu(q+\tfrac{3}{2}) + \mathrm{Re}(r_0).$$

With (4.7) satisfied we have that $\mathcal{E}(\lambda;k)$ exists. The estimate (4.3) need not hold, however. We note that $\mathcal{E}(\lambda;k)$ can be viewed as a Fourier transform with respect to $\log \lambda$. Indeed, we have

$$(4.8) \qquad \lambda^k \mathcal{E}(\lambda;k)$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp\{-iy \log \lambda\} M[H(\pm s);k+iy] M[G(s);1-k-iy]\,dy.$$

Suppose now that (4.4) is satisfied which, in turn, implies that $\varepsilon(k) > 1$. Then by applying the Riemann Lebesgue lemma, we find that, as $\lambda \to \infty$, the right side of (4.8) is $o(1)$, and the estimate (4.3) follows.

Corollary. Let $H(+s)$ satisfy the hypotheses of Lemma 1 and let $G(s)$ satisfy the hypotheses of Lemma 6 with $q = \infty$. Then the infinite expansion

$$(4.9) \quad I^{\pm}(\lambda) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} d_{mn} \lambda^{-(a_m+1)} \sum_{j=0}^{n} \binom{n}{j} (-\log \lambda)^j M^{(n-j)}[H(\pm s);1+a_m]$$

holds as $\lambda \to \infty$.

Proof: It follows from the corollary to Lemma 6, that in this case

$$(4.10)\quad M[H(\pm s);x+iy]M[G(s);1-x-iy] = o(|y|^{-r}), \quad |y| \to \infty,$$

for all r and all x. Hence we can let k go to $+\infty$ in (4.2) and (4.3) to obtain the desired result.

We note that when, in (3.15), $d_{mn} = 0$ for $n \geq 1$ and all m, i.e., when no logarithms appear in the asymptotic expansion of $G(s)$, as $s \to 0+$, the asymptotic expansion (4.9) reduces to

$$(4.11)\qquad I(\lambda) \sim \sum_{m=0}^{\infty} d_{m0}\lambda^{-(a_m+1)} M[H(\pm s);1+a_m].$$

In the proof of Theorem 1, the Riemann Lebesgue lemma was applied to estimate $\lambda^k \mathcal{E}(\lambda;k)$ as $\lambda \to \infty$. Recently, Bleistein, Handelsman and Lew [7], have obtained a generalization of this lemma which, under slightly more restrictive assumptions can be used to improve the error estimate found in Theorem 1. Indeed, we have

Theorem 2. Let the hypotheses of Theorem 1 be satisfied so that (4.5) holds with $\varepsilon(x)$ defined by (4.6). Suppose further that

$$(4.12)\quad M[H(\pm s);x+iy]M[G(s);1-x-iy] \sim c_0 \exp\{i\alpha y^{\mu}\}|y|^{-\varepsilon(x)},$$

as $|y| \to \infty$, where α and μ are any real numbers. Then (4.2) and (4.3) hold with

(4.13) $k < \nu(q+1) + \mathrm{Re}(r_0)$, $k \neq \mathrm{Re}(a_m+1)$, $m = 0,1,2,\ldots.$

Proof: It follows from Theorem 1 that all we need show is that (4.3) holds for all k satisfying (4.13). Thus consider $\lambda^k \mathcal{E}(\lambda;k)$ as given by (4.8). The results of reference [7] show that whenever $\varepsilon(k) > 0$,

$$\lim_{\lambda\to\infty} \lambda^k \mathcal{E}(\lambda;k) = 0, \tag{4.14}$$

except possibly when $1 < \mu < 2$ in (4.12). It is further shown that (4.14) still holds with $1 < \mu < 2$, so long as

$$0 < \varepsilon(k)-(1-\tfrac{\mu}{2}) = q+1-(k-\mathrm{Re}(r_0))/\nu + \tfrac{1}{2}-(1-\tfrac{\mu}{2}). \tag{4.15}$$

Since $1 < \mu$, it is clear (4.15) is satisfied whenever (4.13) holds. This completes the proof.

We wish to emphasize that the hypotheses of Theorem 2 differ from those of Theorem 1 only in that the former includes an additional assumption concerning the oscillatory behavior of $M[H(\pm s);z]M[G(s);1-z]$ in the limit $|y| \to \infty$. It is this more specific information that allows us to apply the results of reference [7] and thereby extend the validity of (4.3) to the region (4.4).

Our concern, of course, is ultimately with the integrals $I_{C_\pm}(\lambda)$ from which the canonical integrals $I^\pm(\lambda)$ were directly derived. We recall that $s = 0$ in

$I^{\pm}(\lambda)$ corresponds to $t = c$ in $I_{c_{\pm}}(\lambda)$ where c is a point in the original domain of integration at which the phase function φ vanishes. If we assume that the conditions of the corollary to Theorem 1 hold, then we find that $s = 0$ is the only critical point for $I^{\pm}(\lambda)$. Thus the infinite expansion (4.9) can be used to obtain the contribution to the asymptotic expansion of (1.2) corresponding to a zero of φ. We must point out, however, that we have not, as yet, established the critical nature of $t = c$. This can be done by explicitly obtaining the expansions of $I_{c_{\pm}}(\lambda)$ and adding them. Only if the resulting sum is non-trivial can we conclude that $t = c$ is a critical point for $I(\lambda)$. We shall investigate this point further in the following section along with some illustrative examples.

5. Explicit Results and Examples.

We wish now to determine the contribution to the asymptotic expansion of (1.2) corresponding to an interior zero of φ. Moreover, we want to express this result explicitly in terms of the original functions φ and g. In principle, we could, by using the results of the previous section, find as many terms of this contribution as desired. The computations, however, become exceedingly awkward as the number of terms increases

and hence, for the most part, we shall be content here with obtaining expansions to leading order only. We shall assume throughout this section that the functions φ and g are sufficiently smooth so that either Theorem 1 or 2 can be applied to obtain the expansions to the orders stated.

Let us suppose that in (1.2) $\varphi(c) = 0$ with $a < c < b$. If, as in Section 2, we denote the contribution corresponding to $t = c$ by $I_c(\lambda)$, then we have

$$I_c = I_{c_+}(\lambda) + I_{c_-}(\lambda) \tag{5.1}$$

with $I_{c_\pm}(\lambda)$ defined by (2.10) and (2.6) respectively.

We now assume that, as $t \to c_+$,

$$\begin{aligned} g(t) &\sim g_+(t-c)^{\mu_+-1}, \quad \varphi(t) \sim r_0(t-c)^{\rho_0} \\ \varphi'(t) &\sim \rho_0 r_0(t-c)^{\rho_0-1}, \quad \rho_0 > 0. \end{aligned} \tag{5.2}$$

In this event, the change of variable (2.9) is easily inverted to leading order. Indeed, we have

$$(t-c) \sim \left(\frac{s}{|r_0|}\right)^{1/\rho_0} \tag{5.3}$$

as $s \to 0+$. Thus, it follows from (2.11) and (5.2) that in (2.10)

$$G_+(s) \sim \frac{g_+}{\rho_0}|r_0|^{-\mu_+/\rho_0} s^{(\mu_+/\rho_0 - 1)} \tag{5.4}$$

as $s \to 0+$. Hence we find from (4.11) that, in this case

$$(5.5)\quad I_{c_+}(\lambda) = \frac{g_+}{\rho_0}\left(\frac{1}{\lambda|\eta_0|}\right)^{\omega_+/\rho_0} M\left[H(\mu_+ s);\frac{\omega_+}{\rho_0}\right] + o\left(\lambda^{-\omega_+/\rho_0}\right),$$

as $\lambda \to \infty$. Here $\mu_+ = \operatorname{sgn} \eta_0$.

Similarly, if as $t \to c_-$

$$(5.6)\qquad \begin{aligned} g(t) &\sim g_-(c-t)^{\omega_- -1}, \qquad \varphi \sim \gamma_0(c-t)^{\nu_0}, \\ \varphi' &\sim -\nu_0\gamma_0(c-t)^{\nu_0 -1}, \qquad \nu_0 > 0. \end{aligned}$$

Then we find

$$(5.7)\quad I_{c_-}(\lambda) = \frac{g_-}{\nu_0}\left(\frac{1}{\lambda|\gamma_0|}\right)^{\omega_-/\nu_0} M\left[H(\mu_- s);\frac{\omega_-}{\nu_0}\right] + o\left(\lambda^{-\omega_-/\nu_0}\right).$$

Here $\mu_- = \operatorname{sgn} \gamma_0$.

Upon adding (5.5) and (5.7) we obtain the desired contribution from $t = c$ to leading order. In most instances the constants in the assumed expansions (5.2) are closely related to the corresponding constants in (5.6). Two cases are worthy of special consideration. Suppose first that g is continuous and non-zero at $t = c$ so that

$$(5.8)\qquad g_+ = g_- = g(c), \qquad \omega_+ = \omega_- = 1.$$

Suppose further that φ is differentiable at $t = c$ with $\varphi'(c) \neq 0$. Then

$$(5.9)\qquad \eta_0 = -\gamma_0 = \varphi'(c), \qquad \rho_0 = \nu_0 = 1.$$

It follows from (5.5-5.9) that

$$(5.10) \qquad I_c(\lambda) \sim \frac{g(c)}{|\varphi'(c)|\lambda}\{M[H(s);1] + M[H(-s);1]\}.$$

Let us now suppose that relations (5.8) hold, but that φ has a simple stationary point at $t = c$. Then

$$(5.11) \qquad r_0 = \gamma_0 = \varphi''(c)/2, \qquad \rho_0 = \nu_0 = 1$$

so that, in this case, we have

$$(5.12) \qquad I_c(\lambda) \sim g(c)\left(\frac{2}{\lambda|\varphi''(c)|}\right)^{1/2} M\left[H(\operatorname{sgn}\varphi''(c)s);\tfrac{1}{2}\right].$$

This last formula is a generalization of the standard stationary phase formula corresponding to $H(s) = \exp(\pm is)$. (See Example 1 below.)

To illustrate what happens when logarithms appear in the expansion of $G(s)$, as $s \to 0+$, let us suppose that $c = a$ in (1.2), φ is as in (5.2), and

$$(5.13) \quad g(t) \sim g_{01}(t-c)^{\omega-1}\log(t-c) + g_{00}(t-c)^{\omega-1}, \quad t \to c_+.$$

After some calculation we find that, in (3.15), $N(0) = 1$ and

$$(5.14) \qquad \begin{aligned} a_0 &= \frac{\omega}{\rho_0} - 1, \qquad d_{01} = \frac{g_{01}}{\rho_0^2}|r_0|^{-\omega/\rho_0}, \\ d_{00} &= \frac{|r_0|^{-\omega/\rho_0}}{\rho_0}\left[g_{00} - \frac{g_{01}\log|r_0|}{\rho_0}\right]. \end{aligned}$$

Thus, it follows from (4.2) and (5.14) that, in this case,

$$I_c(\lambda) = I_{c_+}(\lambda) \sim \frac{(\lambda|\eta_0|)^{-\omega/\eta_0}}{\rho_0}\left\{\frac{g_{01}}{\rho_0}\log\lambda\, M\left[H(\mu_+s);\frac{\omega}{\rho_0}\right]\right.$$

(5.15)

$$\left. + \left(\frac{g_{01}}{\rho_0}\log|\eta_0|-g_{00}\right)M\left[H(\mu_+s);\frac{\omega}{\rho_0}\right] - \frac{g_{01}}{\rho_0}\frac{d}{dz}M[H(\mu_+s);z]\Big|_{z=\frac{\omega}{\rho_0}}\right\}.$$

As a final general result, let us obtain an infinite asymptotic expansion of $I_c(\lambda)$ in the case where

(5.16) $$\varpi(t) \equiv t-c$$

and $g(t)$ is infinitely differentiable at $t = c$. Then

(5.17)

$$g(t) \sim \sum_{m=0}^{\infty} \frac{g^{(m)}(c)}{m!}(t-c)^m, \qquad t \to c_+$$

$$g(t) \sim \sum_{m=0}^{\infty} \frac{(-1)^m g^{(m)}(c)}{m!}(c-t)^m, \qquad t \to c_-.$$

Now upon applying (4.11) we obtain

(5.18) $$I_c(\lambda) \sim \sum_{m=0}^{\infty} \frac{\lambda^{-(m+1)}g^{(m)}(c)}{m!}\{M[H(s);m+1] + (-1)^m M[H(-s);m+1]\}.$$

If any terms in this sum (5.18) are non-zero, then we must conclude that $t = c$ is a critical point for $I(\lambda)$. Alternatively, if the right hand side of (5.18) is identically zero, then $t = c$ is not critical. The issue depends solely on the kernel and at that only through the quantities

(5.19) $$M[H(s);m+1] + (-1)^m M[H(-s);m+1], \quad m = 0,1,2,\ldots.$$

Thus, in general, $t = c$ is critical whenever H is such that at least one of the quantities (5.19) is non-zero. Furthermore, it is readily seen that the same conclusion holds when φ is any C^∞ function that vanishes at $t = c$.

We shall now consider two illustrative examples.

Example 1: Suppose $H(s)$ is the complex Fourier kernel $\exp(is)$. We have by direct computation

$$M[\exp(is);z] = \Gamma(z)\exp\left(\frac{\pi i z}{2}\right) \tag{5.20}$$

and

$$M[\exp(is);z] = e^{-\pi i z}M[\exp(-is);z]. \tag{5.21}$$

From this last relation we find

$$M[\exp(is);m+1] + (-1)^m M[\exp(-is);m+1] = 0, \quad m = 0,1,2,\ldots \tag{5.22}$$

and hence, as anticipated in the introduction, the interior zeros of φ are not critical points for Fourier type integrals.

Suppose now that g is continuous at $t = c$ and φ has a simple stationary point there. Then it follows from (5.12) and (5.20) that

$$I_c(\lambda) \sim g(c)\sqrt{\frac{2\pi}{\lambda|\varphi''(c)|}}\exp\left\{\operatorname{sgn}\varphi''(c)\frac{\pi i}{4}\right\}. \tag{5.23}$$

This will be recognized as the standard stationary phase

formula in the case where $\varphi(c) = 0$. This last restriction is of course unnecessary and can be avoided quite simply. Indeed, suppose that at $t = c$, φ has a simple stationary point, but $\varphi(c) \neq 0$. Then we write

$$I(\lambda) = \exp(i\lambda\varphi(c))\int_a^b \exp[i\lambda(\varphi(t)-\varphi(c)]g(t)dt. \tag{5.24}$$

Since

$$\psi = \varphi(t)-\varphi(c) \tag{5.25}$$

has a simple stationary point at $t = c$ and $\psi(c) = 0$, we find that we need only multiply (5.23) by $\exp\{i\lambda\varphi(c)\}$ to obtain the valid result in this case. Furthermore, the contribution from any critical point at which $\varphi \neq 0$ can be recovered in an analogous manner from the corresponding contribution in the case where φ vanishes at the critical point.

Finally suppose that $c = a$ and (5.13) holds. Suppose further that $\varphi(t)-\varphi(c)$ satisfies the relations satisfied by $\varphi(t)$ in (5.2). Then from (5.15), (5.20) and the remarks of the preceeding paragraph, we find that now

$$\begin{aligned} I_c(\lambda) \sim \exp\left\{i\lambda\varphi(c) + \frac{\mu_+\pi i\omega}{2\rho_0}\right\}\frac{(\lambda|\eta_0|)^{-\omega/\rho_0}}{\rho_0}\Gamma\left(\frac{\omega}{\rho_0}\right) \\ \cdot\left\{\frac{g_{01}}{\rho_0}\log\lambda + \frac{g_{01}}{\rho_0}\log|\eta_0| - g_{00} - \frac{g_0}{\rho_0}\left(\psi\left[\frac{\omega}{\rho_0}\right] + \frac{\mu_+\pi i}{2}\right)\right\}. \end{aligned} \tag{5.26}$$

Here $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$. We might point out that the last case was considered in detail by Erdelyi [8] and by McKenna [9].

Example 2: Let us now suppose that in (1.2)

(5.27) $$H(s) = J_n(s), \quad n = 0,1,2,\ldots.$$

Here J_n is the Bessel function of the first kind of order n. We have [5]

(5.28) $$M[J_n(s);z] = 2^{z-1} \frac{\Gamma(\frac{1}{2}z + \frac{1}{2}n)}{\Gamma(\frac{1}{2}n - \frac{1}{2}z+1)}.$$

Since $J_n(s)$ is even about zero when n is even and odd about zero when n is odd, we have

(5.29) $$\begin{aligned} M[J_n(s);z] &= -M[J_n(-s);z], && n \text{ odd} \\ M[J_n(s);z] &= M[J_n(-s);z], && n \text{ even.} \end{aligned}$$

From this it follows that

(5.30) $$M[J_n(s);m+1] + (-1)^m M[J_n(-s);m+1] = \frac{2^m \Gamma(\frac{1}{2}[m+n+1])}{\Gamma(\frac{1}{2}[n+1-m])}[1+(-1)^{m+n}],$$

$$m = 0,1,2,\ldots$$

and hence for any integer n one half of the quantities (5.30) are not zero. Thus $t = c$ is a critical point in this case.

REFERENCES

[1] Handelsman, R.A. and Lew, J.S.; Arch. Rat. Mech. Anal. 35, 5, (1969).

[2] Handelsman, R.A. and Lew, J.S,; Siam J. Math. Anal. 1, 1, (1970).

[3] Handelsman, R.A. and Lew, J.S.; J.M.A.A. 35, 2, (1971).

[4] Corput, J.C. van Der; Proc. Nederl. Akad. Wet. 51, (1948).

[5] Erdelyi, et.al.; Bateman Manuscript Project, Table of Integral Transforms, Vol. 1, McGraw-Hill, (1954).

[6] Titchmarsh, E.C.; Introduction to the Theory of Fourier Integrals, 2nd Ed., Oxford Univ. Press, London, (1948).

[7] Bleistein, N., Handelsman, R.A. and Lew, J.S.; to appear in SIAM J. Math. Anal.

[8] Erdelyi, A.; SIAM J. on Appl. Math. 4, 1, (1956).

[9] McKenna, J.; SIAM J. on Appl. Math., 15, 4 , (1965).

APPENDIX

In this Appendix, we shall prove Lemmas 1, 2 and 6 of the text.

<u>Proof of Lemma 1</u>: We introduce the functions

$$\sigma_k(s) = \exp\left\{-s^{-k}+is^{\nu}\omega(s)\right\} \sum_{m=0}^{M} \sum_{n=0}^{N(m)} c_{mn} s^{-r_m}(\log s)^n \tag{A1}$$

$$H_k(s) = H(s) - \sigma_k(s). \tag{A2}$$

Here

$$\omega(s) = \sum_{\ell=0}^{\ell<\nu/\delta} b_\ell s^{-\delta\ell} \tag{A3}$$

and for any positive k, we choose $M = M(k)$ to be the largest integer such that

$$\mathrm{Re}(r_M - r_0) < k. \tag{A4}$$

We observe that $H(s)$ and σ_k have identical asymptotic expansions, as $s \to +\infty$, to order $s^{-r_M}(\log s)^{N(M)}$. As a result, $H_k(s) = O(s^{-k+\mathrm{Re}(r_0)})$ from which it follows that $M[H_k(s);z]$ is analytic in a strip with right limit $\mathrm{Re}(z) < k+\mathrm{Re}(r_0)$. The real exponential factor in σ_k assures us that $M[\sigma_k(s);z]$ is analytic in the left half plane $\mathrm{Re}(z) < \mathrm{Re}(r_0)$. Thus the left limit of the strip of analyticity of $M[H_k;z]$ is the same as that of $M[H;z]$. Let us denote this limit by $\mathrm{Re}(z) = \alpha$.

Below, we list the relevant Mellin transforms along with their strips of analyticity. We recall that in its strip of analyticity a Mellin transform decays to zero as $|y| \to \infty$.

Mellin Transform	Strip of Analyticity
$M[H;z]$	$-\alpha < \mathrm{Re}(z) < \mathrm{Re}(r_0)$
$M[\sigma_k;z]$	$-\infty < \mathrm{Re}(z) < \mathrm{Re}(r_0)$
$M[H_k;z]$	$-\alpha < \mathrm{Re}(z) < \mathrm{Re}(r_0) + k$

From the above list we see that in order to analytically continue $M[H;z]$ into the region $\mathrm{Re}(z) < k+\mathrm{Re}(r_0)$ we need only determine the analytic continuation of $M[\sigma_k;z]$ into the strip $\mathrm{Re}(r_0) \leq \mathrm{Re}(z) < \mathrm{Re}(r_0) + k$. Furthermore, since $M[H_k;z]$ decays to zero as $|y| \to \infty$ in $-\alpha < \mathrm{Re}(z) < \mathrm{Re}(r_0) + k$, any algebraic growth, in this limit, of $M[H;z]$ must arise from $M[\sigma_k;z]$.

Since $\sigma_k(s)$ is a finite sum, so is its Mellin transform. A typical term in $M[\sigma_k;z]$ is given by

$$\text{(A5)}\quad I[z;r] = \int_0^\infty \exp\left\{-s^{-k}+is\,\omega(s)\right\}(\log s)^n s^{z-r-1}ds, \quad \mathrm{Re}(z)<\mathrm{Re}(r).$$

In (A5) we rotate the path of integration onto the ray $\arg s = \theta$ where $0 < \theta\,\mathrm{sgn}(b_0) < \pi/2\nu$. The effect of this rotation is to introduce sufficient decay at ∞ so that the integral in (A5) converges for all z. Hence $I[z;r]$ can be continued into the entire z-plane

as a holomorphic function. It remains only to estimate the continuation as $|y| \to \infty$.

In (A5) we stretch the integration variable s by the factor $|y|^{1/\nu}$ to obtain

(A6) $$I[z;r] = |y|^{(x-r)/\nu}\int_0^\infty \exp\left\{-(|y|^{1/\nu}s)^{-k}\right\}s^{x-r-1}$$

$$\cdot\exp\left\{i|y|(s^\nu\omega(s|y|^{1/\nu}) + \operatorname{sgn} y \log s\right\}ds.$$

The function $\omega(s|y|^{1/\nu})$ is a finite sum of the form

(A7) $$\omega(s|y|^{1/\nu}) = b_0 + b_1(s|y|^{1/\nu})^{-\delta} + \cdots.$$

From this we find that the phase function

(A8) $$\psi(s,y) = s^\nu\omega(s|y|^{1/\nu}) + \operatorname{sgn} y \log s$$

has stationary points (points at which $\psi_s = 0$) s_q such that

(A9) $$s_q = \left(\frac{-\operatorname{sgn} y}{\nu b_0}\right)^{1/\nu} + O(|y|^{-\delta}).$$

Here q labels the different choices of the ν^{th} root.

We note that when $-\operatorname{sgn} y/\nu b_0 < 0$, no stationary points are near the positive real axis. When $-\operatorname{sgn} y/\nu b_0 > 0$, however, there are simple stationary points on or near the positive real axis. Armed with this information, we can apply the results of reference [7], to conclude that

$$\text{(A10)} \qquad \lim_{|y|\to\infty} I[z;r] = \begin{cases} O(|y|^{-R}), \quad \text{for all } R; & \dfrac{-\operatorname{sgn} y}{\nu b_0} < 0 \\ O(|y|^{(x-r)/\nu - \frac{1}{2}}); & \dfrac{-\operatorname{sgn} y}{\nu b_0} > 0. \end{cases}$$

This completes the proof.

We remark that the relevant results of reference [7] essentially justify the formal application of the ordinary method of stationary phase to the integral in (A6).

<u>Proof of Lemma 2</u>: By hypothesis

$$\text{(A11)} \qquad M[G;z] = \int_0^\infty s^{iy-1}(s^x G(s))ds$$

is absolutely convergent for all $\operatorname{Re}(z) = x > x_0$.* Upon integrating by parts q times and using the stated properties of $\left(s\frac{d}{ds}\right)^p (s^x G)$, we obtain

$$\text{(A12)} \qquad M[G;z] = \left(\frac{1}{-iy}\right)^q \int_0^\infty s^{iy-1}\left(s\frac{d}{ds}\right)^q (s^x G(s))ds.$$

We now break the interval of integration at the points of discontinuity of $G^{(q+1)}(s)$ and integrate by parts once more in the resulting finite sum of integrals. In this manner we obtain

$$\text{(A13)} \qquad \int_0^\infty s^{iy-1}\left(s\frac{d}{ds}\right)^q (s^x G(s))ds = O(|y|^{-1})$$

*Note that the assumptions made imply that $G(s) = o(s^{-r})$, as $s \to \infty$, for all r and $G(s) = O(s^{-x_0})$ as $s \to 0+$.

which, when combined with (A12), completes the proof.

<u>Proof of Lemma 6</u>: If $\mathrm{Re}(z) = x > -\mathrm{Re}(a_0)$, then the result follows from Lemma 2 when we note that the conditions $\left(s\frac{d}{ds}\right)^p (s^x G) = 0$, $p = 0,1,\ldots,q$, as $s \to 0+$ are implied by the assumed differentiability properties of the expansion (3.15). Now suppose that ρ is any real number greater than $\mathrm{Re}(a_0)$. Also let $\mu(\rho)$ and $\nu(\rho)$ be positive integers satisfying the conditions

$$\text{(A14)} \qquad \mathrm{Re}(a_{\mu-1}) < \rho \le \mathrm{Re}(a_\mu), \qquad \mathrm{Re}(a_0) + \nu > \mathrm{Re}(a_\mu).$$

We now consider the functions

$$\text{(A15)} \qquad \pi_\rho(s) = \left(\sum_{m=0}^{\mu-1} \sum_{n=0}^{N(m)} d_{mn} s^{a_m} (\log s)^n\right) e^{-s^\nu}$$

$$\text{(A16)} \qquad G_\rho(s) = G(s) - \pi_\rho(s)$$

and note that $\nu(\rho)$ has been chosen so that the asymptotic expansions of G and π_ρ agree to order $\mathrm{Re}(a_{\mu-1})$. Thus, as $s \to 0+$

$$\text{(A17)} \qquad G = O\left(s^{\mathrm{Re}(a_\mu)}\right).$$

We also note that $G_\rho(s)$ has all of the properties attributed to $G(s)$ in the statement of the lemma. Hence, upon applying Lemma 2, we immediately find that, for $x > -\mathrm{Re}(a_\mu)$ and $|y| \to \infty$,

$$(A18) \qquad M[G_\rho(s);z] = O(|y|^{-q-1}).$$

By direct calculation we have that

$$(A19) \quad M[\sigma_\rho;z] = \sum_{m=0}^{\mu-1} \sum_{n=0}^{N(m)} d_{mn} \int_0^\infty s^{a_m+z-1} (\log s)^n e^{-s^\nu} ds$$
$$= \sum_{m=0}^{\mu-1} \sum_{n=0}^{N(m)} d_{mn} \frac{d^n}{dz^n}\left(\int_0^\infty s^{a_m+z-1} e^{-s^\nu} ds\right.$$
$$= \sum_{m=0}^{\mu-1} \sum_{n=0}^{N(m)} \frac{d_{mn}}{\nu} \frac{d^n}{dz^n}\left(\Gamma\left[\frac{a_m+z}{\nu}\right]\right)$$

in the region $Re(z) > -Re(a_0)$ and by analytic continuation in the entire z-plane. We know, moreover, that each term in (A19) decays exponentially as $|y| \to \infty$ for <u>all</u> x. Finally, since

$$(A20) \qquad M[G;z] = M[\sigma_\rho;z] + M[G_\rho;z],$$

we have that (3.16) holds for $Re(z) > -Re(a_{u(\rho)})$. However, ρ is arbitrary and $\lim_{\rho\to\infty} Re[a_{u(\rho)}] = \infty$, so that upon letting $\rho \to \infty$ we obtain the desired result.